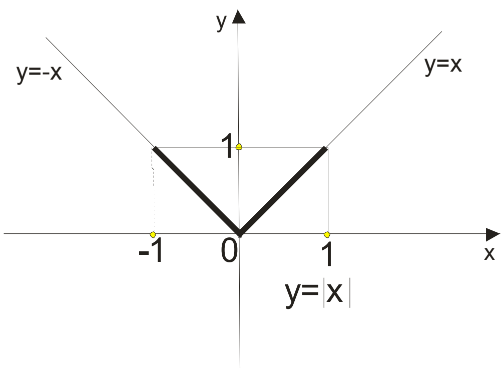
# Fourier series - tasks ( I- part)

**Example 1.**

Function $y = |x|$ developed in Fourier series on the interval $[-\pi, \pi]$

***Solution:***

First, we draw a picture:




Obviously, the function is even (figure is symmetric with respect to the y-axis), and we will use the formula:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \qquad \text{and } b_n = 0$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} xdx = \frac{2}{\pi}\cdot\left(\frac{x^2}{2}\right)\Big/_0^{\pi} = \frac{2}{\pi}\cdot\left(\frac{\pi^2}{2}\right) = \pi$$

Next we ask:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx =$$

This integral will be solved with the help of partial integration, and first without borders…

$$\int x\cos nx dx = \begin{vmatrix} x = u & \cos nx dx = dv \\ dx = du & \frac{1}{n}\sin nx = v \end{vmatrix} = x\cdot\frac{1}{n}\sin nx - \int \frac{1}{n}\sin nx dx = \frac{x\sin nx}{n} - \frac{1}{n}\int \sin nx dx =$$

$$= \frac{x\sin nx}{n} + \frac{1}{n}\frac{1}{n}\cos nx = \frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx$$

Now we put borders:

$$\left(\frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx\right) / \begin{matrix}\pi \\ 0\end{matrix} = \left(\underbrace{\boxed{\frac{\pi\sin n\pi}{n}}}_{\text{this is } 0} + \frac{1}{n^2}\cos n\pi\right) - \left(\frac{0\cdot\sin n\cdot 0}{n} + \frac{1}{n^2}\underbrace{\boxed{\cos n\cdot 0}}_{\text{this is } 1}\right) =$$

$$= \frac{1}{n^2}\cos n\pi - \frac{1}{n^2} = \frac{1}{n^2}(\cos n\pi - 1)$$

Then:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\cdot\frac{1}{n^2}(\cos n\pi - 1) = \frac{2}{\pi n^2}(\cos n\pi - 1)$$

Of course, that $n$ takes values $1,2,3...$

Expression $\cos n\pi$ have values:

for n=1 is $\cos\pi$=-1

for n=2 is $\cos\pi$=1

for n=3 is $\cos\pi$=-1

for n=4 is $\cos\pi$=1

etc.

Therefore, it holds that $\boxed{\cos n\pi = (-1)^n}$

Then is $a_n = \frac{2}{\pi n^2}((-1)^n - 1)$

If n is even number, we have: $a_{2n} = \frac{2}{\pi n^2}((-1)^{2n} - 1) = 0$

If n is odd number, we have: $a_{2n-1} = \frac{2}{\pi(2n-1)^2}((-1)^{2n-1} - 1) = \frac{2}{\pi(2n-1)^2}\cdot(-2) = \frac{-4}{\pi(2n-1)^2}$

Let us return now to the formula for development:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx=\frac{1}{2}\pi+\sum_{n=1}^{\infty}\left(\frac{-4}{\pi(2n-1)^2}\right)\cos(2n-1)x=\frac{1}{2}\pi+\frac{-4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

*So*:

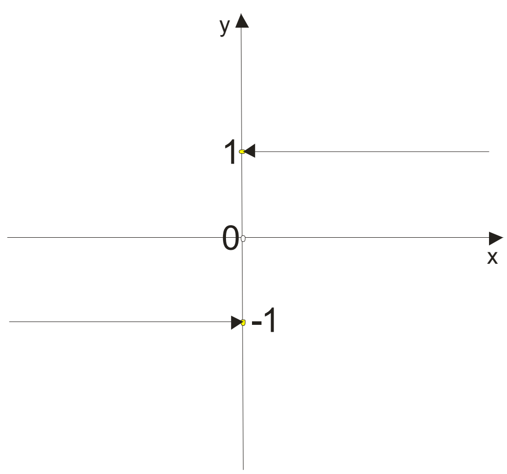
$$\boxed{f(x)=|x|=\frac{1}{2}\pi-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}}$$

**Example 2.**

Developed in Fourier series function $f(x)=\operatorname{sgn} x$ on the interval $[-\pi,\pi]$

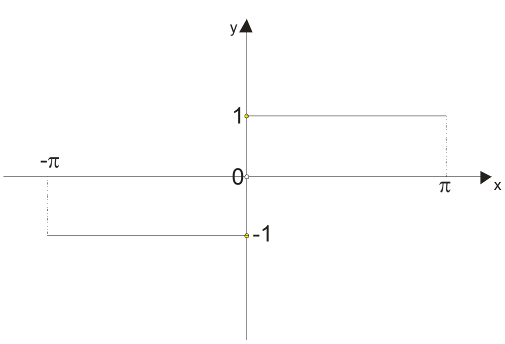
*Solution:*

$$\operatorname{sgn} x=\begin{Bmatrix}-1, \text{ for } x<0\\ 0, \text{ for } x=0\\ +1, \text{ for } x>0\end{Bmatrix}$$ Take a look :




If $x\neq 0$ then we have $\operatorname{sgn} x=\frac{x}{|x|}$

We need this function on the interval $[-\pi,\pi]$:



Obviously, the function is odd, and using the formula : $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$ $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} \sin nx dx = \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)\Big/_{0}^{\pi} = \frac{2}{\pi}\left(-\frac{\cos n\pi}{n} + \frac{\cos n\cdot 0}{n}\right) =$$

$$= \frac{2}{\pi}\left(-\frac{(-1)^n}{n} + \frac{1}{n}\right) = \boxed{\frac{2}{\pi n}(1-(-1)^n)}$$

Again we distinguish between odd and even members:

For n - even number is $\mathrm{b}_{2n} = 0$

for n - odd number is $\mathrm{b}_{2n-1} = \frac{2}{\pi(2n-1)}(1+1) = \frac{4}{\pi(2n-1)}$

Now go back to the starting formula for development and we have:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)}\sin(2n-1)x = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$$

$$\boxed{\operatorname{sgn} x = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}}$$

**Example 3.**

Function $f(x) = \begin{cases} \pi, & -\pi \le x < 0 \\ x, & 0 \le x \le \pi \end{cases}$ develop into a trigonometric series.

*Solution:*

First, we notice that a given interval $[-\pi, \pi]$. This means we will use the formula:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

Beware of one thing: since the function is given in this way we have to work two integrals . In first integral borders are $-\pi$ and 0 and function is *f(x)* = $\pi$, and when borders are 0 and $\pi$ function is *f(x) = x*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi dx + \frac{1}{\pi}\int_0^{\pi} x dx = \frac{1}{\pi}\pi\cdot x\Big/_{-\pi}^{0} + \frac{1}{\pi}\frac{x^2}{2}\Big/_{0}^{\pi} = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\not\pi}\int_{-\pi}^{0} \not\pi \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx$$
$$= \int_{-\pi}^{0} \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx$$

Here we have the integral of the partial integration:

$$\int x\cos nx dx = \left| \begin{matrix} x = u & \cos nx dx = dv \\ dx = du & \frac{1}{n}\sin nx = v \end{matrix} \right| = x\cdot\frac{1}{n}\sin nx - \int \frac{1}{n}\sin nx dx = \frac{x\sin nx}{n} - \frac{1}{n}\int \sin nx dx =$$
$$= \frac{x\sin nx}{n} + \frac{1}{n}\frac{1}{n}\cos nx = \frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx$$

Now we return to $a_n$:

$$a_n = \int_{-\pi}^{0} \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx = \frac{1}{n}\sin nx \Big/ {}^{0}_{-\pi} + \frac{1}{\pi}\left(\frac{x\sin nx}{n} + \frac{1}{n^2}\cos nx\right) \Big/ {}^{\pi}_{0}$$
$$= [\underbrace{\frac{1}{n}\sin n\cdot 0 - \frac{1}{n}\sin n(-\pi)}_{\text{this is all } 0}] + \frac{1}{\pi}[\left(\underbrace{\frac{\pi\sin n\pi}{n}}_{\text{this is } 0} + \frac{1}{n^2}\cos n\pi\right) - \left(\underbrace{\frac{0\sin n\cdot 0}{n}}_{\text{this is } 0} + \frac{1}{n^2}\cos n\cdot 0\right)]$$
$$= \frac{1}{\pi}\left(\frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right) = \frac{1}{\pi n^2}(\cos n\pi - 1) = \frac{1}{\pi n^2}((-1)^n - 1)$$

*So*:

$$a_n = \frac{1}{\pi n^2}((-1)^n - 1) = \begin{cases} \frac{-2}{\pi(2k+1)^2}, & n = 2k+1, \quad k = 0,1,2,3... \\ 0, & n = 2k, \quad k = 0,1,2,3... \end{cases}$$

More to find:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_{-\pi}^{0} \pi\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx dx$$
$$= \int_{-\pi}^{0} \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx dx$$

And here we will first do the partial integration:

$$\int x\sin nx dx = \left| \begin{matrix} x = u & \sin nx dx = dv \\ dx = du & -\frac{1}{n}\cos nx = v \end{matrix} \right| = -x\cdot\frac{1}{n}\cos nx + \int \frac{1}{n}\cos nx dx = -\frac{x\cos nx}{n} + \frac{1}{n}\int \cos nx dx =$$
$$= -\frac{x\cos nx}{n} + \frac{1}{n}\frac{1}{n}\sin nx = \boxed{-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx}$$

Now we have :

$$b_n = \int_{-\pi}^{0} \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} x \sin nx dx$$

$$= \left(-\frac{1}{n}\cos nx\right) / {0 \atop -\pi} + \frac{1}{\pi}\left(-\frac{x\cos nx}{n} + \frac{1}{n^2}\sin nx\right) / {\pi \atop 0}$$

$$= \left\{\left(-\frac{1}{n}\cos n\cdot 0\right) - \left(-\frac{1}{n}\cos n(-\pi)\right)\right\} + \frac{1}{\pi}\left\{\left(-\frac{\pi\cos n\pi}{n} + \frac{1}{n^2}\sin n\pi\right) - \left(-\frac{0\cdot\cos(n\cdot 0)}{n} + \frac{1}{n^2}\sin(n\cdot 0)\right)\right\}$$

$$= -\frac{1}{n} + \frac{1}{n}\cos n\pi + \frac{1}{\pi}\left(-\frac{\pi\cos n\pi}{n}\right)$$

$$b_n = -\frac{1}{n} + \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n}$$

$$b_n = -\frac{1}{n}$$

Now we can write the entire development:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\boxed{f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\cos((2k+1)\pi)}{(2k+1)^2} - \sum_{n=1}^{\infty}\frac{\sin nx}{n}}$$

This series converges to the function S which, according to Dirihleovoj theorem coincides with the function f on the interval:

$[-\pi, 0) \cup (0, \pi]$ and because f(x) has interruption for x = 0 , then $S(0) = \frac{f(0-0)+f(0+0)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$

graph see in the picture: